ON THE INTEGRAL EQUATION

$$\lambda \, f(x) = \int_0^a e^{-(x-y)^2} f(y)dy$$

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§1. Introduction.

We wish to consider briefly the integral equation

$$(1) \qquad \lambda f(x) = \int_0^a K(x - y)f(y)dy, \qquad a > 0,$$

which occurs in connection with various problems of probability theory and mathematical physics. Unless $K(x)$ is a function of particularly simple type, such as a polynomial or sum of exponentials, the problem of obtaining an exact solution of (1) and of determining the characteristic values seems exceedingly difficult. In the present note, we discuss the behavior of the largest characteristic value, λ_M, as a $\longrightarrow \infty$, under certain assumptions concerning $K(x)$, and present a method for obtaining λ_M when a is not large.

Our first result is

Theorem 1. If

$$(2) \qquad \text{(a)} \quad K(x) \text{ is non-negative, even and monotone}$$
$$\text{decreasing for } 0 \leq x < \infty,$$

$$\text{(b)} \quad c = \int_0^\infty K(x)dx < \infty,$$

then as a $\longrightarrow \infty$, $\lambda_M \longrightarrow 2c$.

More precisely, for all $a > 0$,

$$(3) \qquad 2 \int_0^{a/2} K(x)dx \geq \lambda_M \geq 2 \int_0^a K(x)dx - \frac{2}{a} \int_0^a xK(x)dx.$$

We give two proofs of the result that $\lambda_M \longrightarrow 2c$, the first depending upon variational principles and the second upon an important property of the characteristic function associated with λ_M.

§2. First proof.

We employ the following two lemmas, the first of which is well-known:

Lemma 1.

$$(1) \qquad \lambda_M = \underset{f}{\text{Max}} \ \frac{\int_0^a \int_0^a K(x - y)f(x)f(y)dx\ dy}{\int_0^a f^2(x)dx}$$

Lemma 2. **If** $K(x, y) \geq 0$ **in** $0 \leq x, y \leq a$, **and** λ_M **denotes as above the largest characteristic root of** $K(x, y)$, **then**

$$(2) \qquad \underset{g \geq 1}{\text{Sup}} \ \underset{x}{\text{Min}} \ \frac{\int_0^a K(x, y) g(y)dy}{g(x)} \leq \lambda_M \leq \underset{g \geq 1}{\text{Inf}} \ \underset{x}{\text{Max}} \ \frac{\int_0^a K(x, y)g(y)dy}{g(x)}$$

Proof: As is known, the characteristic function $f(x)$ associated with λ_M is positive, by virtue of the non-negativity of $K(x, y)$. Let $g(x)$ be a positive function greater than or equal to one. From

(3) $\qquad \lambda_M f(x) = \int_0^a K(x, y) f(y) dy,$

we obtain

(4) $\qquad \lambda_M \int_0^a f(x) g(x) dx = \int_0^a \left(\int_0^a K(x, y) g(y) dy \right) f(y) dy$

$$= \int_0^a \frac{\left(\int_0^a K(x, y) g(y) dy \right)}{g(y)} f(y) g(y) dy,$$

whence (2) follows immediately.

That the two sides of the inequality in (2) are actually equal
and equal to λ_M is a result of Bohnenblust. Extensions of this
result will be found in a paper, soon to appear, by Bohnenblust and
Karlin [2], and applications of this method will be found in a paper
by Bellman and Harris, [1].

Lemma 1 contains the essence of the Rayleigh—Ritz method, and
furnishes lower bounds for λ_M. Lemma 2, which is also based upon
variational principles, furnishes upper and lower bounds. Combining
the two, we obtain

(5) $\qquad \underset{g \quad 0 \le x \le a}{\text{Inf} \ \text{Max}} \ \frac{\int_0^a K(x-y) g(y) dy}{g(x)} \ge \lambda_M$

$$= \underset{f}{\text{Max}} \ \frac{\int_0^a \int_0^a K(x-y) f(x) f(y) dx\, dy}{\int_0^a f^2(x) dx} \ge \underset{g \ge 1 \quad 0 \le x \le a}{\text{Sup} \ \text{Min}} \ \frac{\int_0^a K(x-y) g(y) dy}{g(y)}$$

The simplest possible choices of $f(x)$ and $g(x)$, $f = g = 1$, yield (3) of §1. It is possible that these results may be further refined by a cleverer choice of $f(x)$ and $g(x)$. However, the calculations rapidly become complicated.

Setting $f = 1$, we have

$$(6) \qquad \lambda_M \geq \frac{\int_0^a \left[\int_0^a K(x-y)dy\right]dx}{a}$$

$$\geq \frac{1}{a}\int_0^a\left[\int_0^x K(u)du + \int_0^{a-x}K(u)du\right]dx = \frac{2}{a}\int_0^a\left[\int_0^x K(u)du\right]dx.$$

Integration by parts yields

$$(7) \qquad \lambda_M \geq 2\int_0^a K(u)du - \frac{2}{a}\int_0^a uK(u)du.$$

Setting $g = 1$, we have

$$(8) \qquad \underset{0 \leq x \leq a}{\text{Max}} \int_0^a K(x-y)dy \geq \lambda_M.$$

Since K is even and monotone decreasing, it is easily seen that the maximum occurs at $x = a/2$. Thus,

$$(9) \qquad \int_0^a K\left(\frac{a}{2}-y\right)dy = 2\int_0^{a/2}K(y)dy \geq \lambda_M.$$

If $\int_0^\infty K(x)dx < \infty$, it follows readily that $\int_0^a xK(x)dx = o(a)$ as $a \longrightarrow \infty$, and thus that $\lambda_M \longrightarrow 2\int_0^\infty K(u)du$ as $a \longrightarrow \infty$.

The bounds for λ_M will only be narrow for fairly large a, the magnitude depending upon K(x). Taking the interesting case $K(x) = e^{-x^2}$, we obtain

$$(10) \qquad 2\int_0^{a/2} e^{-x^2}dx \geq \lambda_M \geq 2\int_0^a e^{-x^2}dx - \frac{1}{a} + \frac{e^{-a^2}}{a},$$

which yields the results

$$(11) \qquad .843 \geq \lambda_M(2)/\sqrt{\pi} \geq .713$$

$$.995 \geq \lambda_M(4)/\sqrt{\pi} \geq .749$$

$$.999 \geq \lambda_M(10)/\sqrt{\pi} \geq .899$$

We see that even for small a, (10) yields a rough idea of the true value of λ_M.

§3. Second Proof.

The second method of proof yields the following useful result:

Theorem 2. If K(x) is non-negative, even and monotone decreasing for $0 \leq x < \infty$, the characteristic function $f_M(x)$ associated with λ_M, which we normalize by the requirement $\int_0^a f_M(x)dx = 1$, possesses the following properties:

(1)   (a) $f_M(x) = f_M(a - x),$

     (b) $f_M(x)$ is monotone increasing in $0 \leq x \leq \frac{a}{2}$.

Proof: We require the following two lemmas, the first of which is a well-known result in the theory of integral equations.

Lemma 3. Let $K(x, y)$ be a continuous symmetric function defined over the square $0 \leq x, y \leq a$, and $g(x)$ be continuous over $0 \leq x \leq a$ and not identically zero. Then, if we define

$$(1) \qquad Tg = \int_0^a K(x, y)g(y)dy,$$

the limit

$$(2) \qquad \lim_{n \to \infty} \frac{T^n g}{\lambda_M^n} = \phi(x)$$

exists and is a characteristic function of $K(x, y)$ associated with λ_M.

Lemma 4. If $f(x)$ has the following properties:

$$(3) \quad (a) \qquad f(x) = f(a - x)$$

$$\quad (b) \qquad f'(x) \geq 0 \quad \text{for} \quad 0 \leq x \leq \frac{a}{2},$$

then

$$(4) \qquad Tf = \int_0^a K(x - y)f(y)dy$$

possesses the same properties, provided that $K(x)$ is even and monotone decreasing in the interval $[0, a]$.

Proof of Lemma 4: We have

(5) $g(x) = Tf = 2 \int_0^{a/2} [K(x - y) + K(a - x - y)] f(y) dy,$

whence

(6) $g'(x) = 2 \int_0^{a/2} [K'(x - y) - K'(a - x - y)] f(y) dy.$

Integration by parts yields

(7) $g'(x) = 2f(0) [K(x) - K(a-x)] + 2 \int_0^{a/2} [K(x-y) - K(a-x-y)] f'(y) dy.$

If $0 \leq x, y \leq a/2$, we have $x \leq a - x$, $|x - y| \leq a - x - y$, and consequently $K(x) \geq K(a - x)$, $K(x - y) \geq K(a - x - y)$. Therefore $g'(x) \geq 0$, with equality only at $x = a/2$.

We now combine Lemmas 3 and 4 to prove Theorem 2. Let $f_0 = 1$ and define

(8) $f_{n+1}(x) = \int_0^a K(x - y) f_n(y) dy.$

From Lemma 2, it follows that each $f_n(x)$ possesses properties 3a and 3b since f_0 does trivially. Lemma 3 shows that

(9) $f(x) = \lim_{n \to \infty} f_n(x) / \lambda_M^n$

is a characteristic function of $K(x - y)$ associated with λ_M. It follows from Lemma 4 that $\phi(x)$ possesses properties 1a and 1b.

The monotonicity property of $f(x)$ will play an important role in our second proof of Theorem 1. Let us normalize our solution,

which we know is positive, by the condition

(10) $$\int_0^a f(x)dx = 1.$$

Integrating both sides of (1) of §1 between 0 and a we obtain

(11) $$\lambda_M = \int_0^a \left[\int_0^a K(x-y)dx\right] f(y)dy$$

$$= 2\int_0^{a/2} \left[\int_0^y K(u)du + \int_0^{a-y} K(u)du\right] f(y)dy.$$

In $\left[0, \frac{a}{2}\right]$, we have

(12) $$\left|\int_0^{a-y} K(u)du - c\right| \leq \int_{a/2}^{\infty} K(u)du.$$

Let Y be taken between 0 and a/2. Then, we obtain after some simplification,

(13) $$\left|\lambda_M - 2\int_Y^a \left[2c - \int_Y^{\infty} K(u)du\right] f(y)dy\right| \leq \int_{a/2}^{\infty} K(u)du + 4c\int_0^Y f(y)dy,$$

or

(14) $$|\lambda_M - 2c| \leq \int_{a/2}^{\infty} K(u)du + 8c\int_0^Y f(y)dy + \int_Y^{\infty} K(u)du.$$

It remains to choose Y advantageously and estimate $\int_0^Y f(y)dy$. We have for $0 \leq y \leq a/2$, using the monotonic character of $f(x)$,

(15) $$\frac{1}{2} = \int_0^{a/2} f(x)dx \geq \int_y^{a/2} f(x)dx \geq f(y)\left(\frac{a}{2} - y\right),$$

and thus $f(y) \leq 1/(a - 2y)$. Hence

$$(16) \qquad \int_0^Y f(y)dy \leq Y/(a - 2Y).$$

If $Y \longrightarrow a$, $Y/a \longrightarrow 0$ as $a \longrightarrow \alpha$, we see that $\lambda_M \longrightarrow 2c$. Choosing Y so that $8cY/(a - 2Y) = \int_Y^\infty K(u)du$, we obtain a best possible error term. For example, if $K(x) = e^{-x^2}$, we obtain in this manner, as $a \longrightarrow \infty$,

$$(17) \qquad | \lambda_M - 2c | = 0\left(\frac{\sqrt{\log a}}{a}\right),$$

which is inferior to the result stated in Theorem 1.

§4. An Approximation Method for Small a.

Referring to (5) of §2, we see that it is possible to improve our estimates for λ_M by choosing in place of $f = g = 1$, functions which more nearly represent $f_M(x)$. Since we know the general form of $f_M(x)$ from Theorem 2, it would seem that two classes of functions which might yield good results are given by

$$(1) \qquad f(x) = 1 + cx(a - x), \qquad c \geq 0,$$

and

$$(2) \qquad f(x) = 1, \qquad 0 \leq x \leq b < \frac{a}{2},$$

$$= c, \qquad b \leq x \leq a - b,$$

$$= 1, \qquad a - b \leq x \leq a, \qquad c \geq 1.$$

If we are concerned with $K(x) = e^{-x^2}$, in each of these cases the numerical work will not be too complicated, since the integrals that occur can be evaluated in terms of tabulated functions.

For a general $K(x)$, the upper limit can be evaluated in terms of $\int_0^x K(y)dy$, if we use the second class of functions, the step-functions. These constitute a 2-parameter family with b and c free to be varied.

REFERENCES

[1] R. Bellman and T. E. Harris, Positive Transformations, Expectations, and Tauberian Theorems, to appear.

[2] F. Bohnenblust and S. Karlin, Positive Operators, to appear.